THE EFFECTS OF VISCOSITY AND HEAT CONDUCTION WITH WEAK MACH REFLECTION

PMM Vol. 33, №2, 1969, pp. 368-375

G. P. SHINDIAPIN
(Saratov)

(Received July 12, 1968)

Propagation of a shock wave along a broken wall can involve a complex pattern of intersection of three shock waves (the incident wave, the reflected wave, and the Mach wave). Such a pattern is known as "irregular" or "Mach" reflection. Neumann [1] was the first author to investigate the combination of three coincident plane jumps (the three-jump theory) and to note the discrepancy between theoretical and experimental results. This discrepancy is especially marked in the case of weak shock waves for which the experimental values of the shock-wave angles at the triple point lie in a range where the three-jump problem has no solution. A major role in the establishment of this fact was played by Smith's experimental findings [2] which were later refined in [3 and 4].

The discrepancy for weak Mach reflection became widely known with the appearance of surveys [5 and 6].

Noteworthy attempts to eliminate the discrepancy between theory and experiment and to explain the shortcomings of the three-jump theory were made by Sternberg [7] and Sakurai [8]. Analyzing the boundary conditions used in three-jump interaction, Sternberg concluded that the simple conditions of equal pressure and direction beyond the triple point are not valid for a real fluid of finite viscosity. Nonviscous gas theory implies unrestricted curving of the shock wave near the triple point, which conflicts with the existence of the Rankine-Hugoniot jump. Sternberg suggested that viscosity alters the boundary conditions at the triple intersection and postulated the existence of a bounded (non-Hugoniot) shock-wave region changes the structure of the shock wave above and below the triple point. Although this region is itself small, it can have a considerable effect on the downstream flow.

Sakurai carried out a quantitative analysis of the boundary conditions of the three-jump theory and compared theoretical and experimental values of the angles. This led him to conclude that an essentially inhomogeneous flow zone where viscosity plays an important role exists beyond the triple point, and that this requires the use of the Navier-Stokes equations in constructing the solution. Sakurai's solutions [8], which are the first approximations in the series expansions of the required functions about the triple point, establish a correspondence between the theoretical and experimental values of the shock-wave angles, but afford no clear notion of the effect of viscosity on the character of flow beyond the triple point. The numerical results recently obtained by Shao for the Mach reflection of viscous condensation jumps are also of interest.

Experimental studies and numerical calculations show that a fairly abrupt change in the flow parameters occurs downstream from the triple point, and that this change occurs in a bounded zone adjacent to the shock fronts. Such flows are called "short waves". Their general theory for an ideal gas is developed in [10]. The short-wave type equations for a viscous gas were first derived by Ryzhov and Shefter [11] and are simplifications of the Navier-Stokes equations for describing two-dimensional unsteady flows. They are similar to the simplifications worked out for steady transonic gas flows by Sichel [12], Ryzhov [13], and other authors.

In the present paper we obtain a solution of the system of viscous short-wave equations similar to the solution of [14] which enables us to investigate the flow downstream from the triple point. Mathematically, the problem reduces to the solution of a third-order ordinary differential equation under conditions corresponding to the boundary conditions of the short-wave zone. The dependence of the solution on the parameter which includes the viscosity and heat conduction coefficients enables us to trace the effect of dissipative factors on the magnitude and structure of the inhomogeneous stream region beyond the triple point. We investigate the solution for the limiting case of viscosity, i.e. as $\lambda \to 0$. Arguments indicating that the solution for $\lambda \to 0$ is disctinct from the solution for $\lambda = 0$ are cited.

1. Let us derive the system of short-wave equations for a viscous heat-conducting gas which describe the flow structure in the neighborhood of the triple point. Our derivation follows that of Ryzhov and Shefter [11]. The continuity, Navier-Stokes, energy, and state equations in the case of two-dimensional unsteady flows can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y} = 0 \tag{1.1}$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\eta_1\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left[2\eta_1\frac{\partial v_x}{\partial x} + (\eta_2 - {}^2/_3\,\eta_1)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] \tag{1.2}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\eta_1\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\eta_1\frac{\partial v_y}{\partial y} + (\eta_2 - {}^2/_3\,\eta_1)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] \tag{1.3}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y}\right) - \alpha T\left(\frac{\partial p}{\partial t} + v_x\frac{\partial p}{\partial x} + v_y\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varkappa\frac{\partial T}{\partial y}\right) + (\eta_2 - {}^2/_3\,\eta_1)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)^2 + 2\eta_1\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2\right\} \tag{1.4}$$

$$n\,dp = a^2 d\rho + \alpha\rho a^2 dT, \qquad \alpha^2 a^2 T = (n-1)\, c_p \tag{1.5}$$

Here x, y are orthogonal Cartesian coordinates; t is the time; v_x, v_y are the components of the velocity vector ; ρ, p, T are the density, pressure, and temperature; a is the velocity of sound; α is the coefficient of thermal expansion; n is the ratio of specific heats; η_1, η_2, $\varkappa$ are the coefficients of viscosity, second viscosity, and thermal conductivity. It is also convenient to introduce the coefficent of "longitudinal" viscosity $\eta = {}^4/_3\,\eta_1 + \eta_2$.

Let a wave propagate in the direction of the x-axis in an unperturbed gas with the parameters p_0, ρ_0, T_0, . . ., η_0 ; let the excess quantities in the wave be small as compared with the initial values, and let the pressure, density, temperature, and velocity-of-sound perturbations be of the same order of smallness as the longitudinal component v_x of the velocity vector.

As in the short-wave theory of an ideal gas [10], we introduce a moving coordinate system and convert to dimensionless variables,

$$\frac{x}{a_0 t} = 1 + LX, \qquad \frac{y}{a_0 t} = \frac{L}{\varepsilon} Y, \qquad t = \frac{\eta_0}{\rho_0 a_0^2} \frac{\tau}{\Delta}$$

$$v_x = a_0 N V_X, \quad v_y = a_0 N \varepsilon V_Y, \quad p = \rho_0 a_0^2 (p_0 / \rho_0 a_0^2 + NP')$$

$$\rho = \rho_0 (1 + NR), \qquad T = T_0 (1 + N\theta), \qquad a = a_0 (1 + NA) \tag{1.6}$$

Here the dimensionless quantities X, Y, τ, V_X, V_Y, P', R, θ, A are of the order of unity, and L, ε, Δ, N are small as compared with unity.

Let us divide the coefficients of viscosity and thermal conductivity (which are of the same order of magnitude [15]) by η_0, and the remaining coefficients by their values in the equilibrium state. Denoting small perturbations by primes, we can write

$$\eta_1 = \eta_0 (\eta_{01} / \eta_0 + \eta_1'), \ldots, \alpha = \alpha_0 (1 + \alpha') \tag{1.7}$$

Let us substitute variables (1.6), (1.7) into system (1.1)–(1.5) and retain only the leading terms in the resulting expressions.

In the case of two-dimensional flows ($N = L$, $\varepsilon = L^{1/2}$, $\Delta = L^2$) with slow variation of the wave parameters with time ($\partial (\ldots) / \partial\tau \ll 1$) we have quasisteadystate flows; in terms of the variables

$$X' = X, \quad Y' = \sqrt{2}\, Y, \qquad V'_X = m_0 V_X, \qquad V'_Y = {}^1/_2 \sqrt{2}\, m_0 V_Y$$

$$P = \frac{\eta_0 c_{p_0}}{\varkappa_0}, \quad \tau' = \frac{2\tau}{1 + (n_0 - 1) / P}, \quad m = \frac{1}{2\rho^3 a^2} \left[\frac{\partial^2 p}{\partial (1/\rho)^2} \right]_S \tag{1.8}$$

we arrive at quasiselfsimilar flows described by a system of short waves (the primes are omitted),
$$m_0 R = V_X, \qquad m_0 P' = V_X, \qquad m_0 \theta = [(n_0 - 1) / \alpha_0 T_0]\, V_X$$

$$\frac{\partial V_X}{\partial X} = \frac{\partial V_Y}{\partial Y}, \qquad (V_X - X) \frac{\partial V_X}{\partial X} - Y \frac{\partial V_X}{\partial Y} + \frac{\partial V_Y}{\partial Y} - \frac{1}{\tau} \frac{\partial^2 V_X}{\partial X^2} = 0 \tag{1.9}$$

Although the coefficients of viscosity do not appear in (1.9), their order is allowed for by means of the quantity Δ, and their ratio determines the Prandtl number P; the number m_0 is the value of m in the unperturbed gas. In the case of a perfect gas (i.e. one which conforms to the Clapeyron equation of state) $m_0 = {}^1/_2 (n_0 + 1)$ and τ plays the role of a parameter.

2. In order to investigate the flow structure in the short-wave zone we introduce the cylindrical coordinate system r, ϑ, t and the corresponding moving system δ, Y, τ. The velocity components and coordinates are related by the expressions

$$\mu = \frac{M}{M_0} = \frac{1}{M_0} \frac{u}{a_0}, \qquad \nu = \frac{1}{M_0 \sqrt{m_0 M_0}} \frac{v}{a_0}, \qquad \delta = \frac{1}{m_0 M_0} \left(\frac{r}{a_0 t} - 1 \right)$$
$$Y = \vartheta / \sqrt{m_0 M_0}, \qquad \tau = \ln t, \qquad M_0 = N / m_0 \tag{2.1}$$

Conversion from the variables of moving Cartesian system (1.8) to the variables of moving cylindrical system (2.1) by means of the equations

$$V_X' = \mu, \qquad V_Y' = {}^1/_2 / \sqrt{2} (\mu Y + \nu), \qquad X' = \delta - {}^1/_2 Y^2, \qquad Y' = \sqrt{2} Y$$
$$\ln c\tau' = \tau, \quad c = \eta_0 [1 + (n_0 - 1) / P]^{-1} [2\rho_0 a_0^2 M_0^2 m_0^2]^{-1} \quad (\delta = X + {}^1/_2 Y^2) \tag{2.2}$$

transforms system (1.9) into

$$\frac{\partial \nu}{\partial \delta} = \frac{\partial \mu}{\partial Y}, \qquad (\mu - \delta) \frac{\partial \mu}{\partial \delta} + \frac{1}{2} \frac{\partial \nu}{\partial Y} + \frac{1}{2} \mu - \lambda \frac{\partial^2 \mu}{\partial \delta^2} = 0 \tag{2.3}$$

Here $\lambda = c / t$ and for $\lambda = 0$ $(\eta_0 = 0)$ system (2.3) corresponds to the short-wave system for an ideal gas [10].

The first three equations of (1.9) are the integrals of the system and can be expressed in terms of variables (2.1) as follows:

$$M = \frac{p - p_0}{n_0 p_0} = \frac{\rho - \rho_0}{\rho_0} = \frac{\alpha_0 T_0}{n_0 - 1} \frac{T - T_0}{T_0} \tag{2.4}$$

This means that the motion of the gas is adiabatic and (by virtue of the first equation of (2.3)) nonvortical in our approximation.

3. Eliminating the function v from the equations of system (2.3), we arrive at the equation

$$\mu_\delta^2 + (\mu - \delta)\mu_{\delta\delta} + {}^1/_2\mu_{YY} + {}^1/_2\mu_\delta - \lambda\mu_{\delta\delta\delta} = 0 \tag{3.1}$$

which the substitution $\mu = \delta + R$ reduces to the form

$$(R^2)_{\delta\delta} + R_{XX} + 3R_\delta + 1 - 2\lambda R_{\delta\delta\delta} = 0 \tag{3.2}$$

The subscripts denote partial derivatives with respect to the indicated variables.

A transformation of the Tomotika-Tamada type [14],

$$R = Z(S) - 2\sigma^2 Y^2, \qquad S = \delta + \sigma Y^2 \tag{3.3}$$

reduces Eq. (3.2) to a nonlinear ordinary differential equation of the form

$$-2\lambda Z''' + (Z^2)'' + (2\sigma + 3) Z' - 4\sigma^2 + 1 = 0 \tag{3.4}$$

or

$$-\lambda Z''' + ZZ'' + (Z' + 2\sigma + 1)(Z' - \sigma + {}^1/_2) = 0 \tag{3.5}$$

4. We shall attempt to find the solution of (3.4) which in the limiting case has as its equivelocity line $\mu_1 = \mu(\delta, Y)$ the sonic line $\delta = \mu_1$, where the velocity gradients (and, by (2.4), the pressure, density, and temperature gradients) vanish, i.e.

$$\mu(\delta, Y) = \mu_1, \quad \mu_\delta(\delta, Y) = 0,$$
$$\mu_Y(\delta, Y) = 0 \quad \text{for} \quad \delta = \mu_1 \tag{4.1}$$

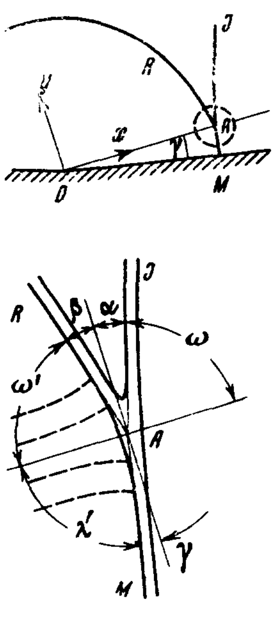

Fig. 1

This is a necessary condition in the case of nonviscous flow in the neighborhood of the triple point, i.e. when the reflected wave degenerates into the sonic line [16].

By virtue of (3.1), (3.3) conditions (4.1) are fulfilled for

$$Z|_{\delta=\mu_1} = 2\sigma^2 Y^2, \qquad Z'|_{\delta=\mu_1} = -1 \tag{4.2}$$

and for the values $\sigma = 0$, $\sigma = -{}^1/_2$.

The constant c in the equation

$$(Z^2)' + (2\sigma + 3) Z - (4\sigma^2 - 1) S + c = 0 \tag{4.3}$$

obtained by lowering the order in (3.4) assumes the values $c = -\mu_1$, $c = 0$, respectively. The solutions of (4.3) which satisfy conditions (4.1) are of the form

$$Z = \mu_1 - \delta \quad \text{for} \quad \sigma = 0 \tag{4.4}$$

$$Z = \mu_1 - \delta + {}^1/_2 Y^2 \quad \text{for} \quad \sigma = -{}^1/_2 \tag{4.5}$$

It is clear that solutions (4.4), (4.5) belong to the classes of special solutions of Eq. (3.5), i.e. that

$$Z = -(2\sigma + 1) S + C \quad \text{for} \quad \sigma = 0 \tag{4.6}$$

$$Z = (\sigma - {}^1/_2) S + C \quad \text{for} \quad \sigma = -{}^1/_2$$

and are therefore the solutions of the viscous equation.

Further on we shall show that the physical flow pattern in the neighborhood of the triple point is associated with the value $\sigma = 0$.

5. Let us consider the flow of a viscous gas beyond the triple shock-wave configuration which arises with Mach reflection of a weak shock wave from a broken rigid wall (Fig. 1).

We assume (in accordance with Sternberg's model [7]) that the effects of viscosity and thermal conductivity are substantial beyond the transitional (non-Hugoniot) zone separating the three shock waves at which the usual conditions of passage through a shock-wave front are fulfilled. We shall describe the flow beyond the transition zone by means of the short-wave model, requiring that the solutions satisfy the usual conditions of passage through shock-wave fronts outside the transition region (i.e. above and below the triple point). Moreover, we require that the short-wave region be closed downstream by a boundary where the gradients of all the stream parametrs vanish.

The intensity M_1 and the inclination ω of the incident wave AJ are given in terms of, and determined by, the intensity and inclination ω', λ' of the reflected wave and the Mach wave (far away from A).

The conditions at the shock fronts for the normal and tangent velocity components in the variables of cylindrical system (2.1) are of the form

$$\mu = \frac{M}{M_0} = \frac{1}{M_0}\,\frac{p - p_0}{n_0 p_0}, \qquad (\mu - \mu')\,\psi^\circ - \nu = \mu'\,(\alpha^{\circ\prime} + Y)$$
$$\alpha^{\circ\prime} = \alpha' / \sqrt{m_0 M_0}, \quad \psi^\circ = \psi / \sqrt{m_0 M_0} \tag{5.1}$$

Here μ' is the velocity component behind the wave front; α' is the angle between the velocity vector behind the front and the axis $\vartheta = 0$; ψ is the angle between the radius vector and the normal to the wave front. From now on we shall assume that $m_0 = {}^1/_2\,(n_0 + 1)$, that the gas is perfect, and that the wave front is described by the equation

$$\frac{d\delta}{dY} = \psi^\circ = \pm \sqrt{2\delta - (\mu + \mu')} \tag{5.2}$$

Beyond the incident wave front and outside the interaction zone we have a homogeneous stream with the velocity $\mu_1 = M_1 / M_0$. The velocity (pressure) beyond the reflected wave front decreases with distance from the triple point, and at some point R lying on the boundary which closes the short-wave zone we have $\mu_R \doteq \mu_1$. At this point the reflected front degenerates into the sonic line $\delta = \mu_1$ in accordance with (5.2); conversely, beyond the Mach wave front the velocity (pressure) increases with decreasing distance from the wall (more precisely, from the outer limit of the boundary layer at the wall). Taking the value of M at the base of the Mach stem as our M_0, we obtain $\mu_0 = 1$.

Let us find the coordinates of the points R, A, M and determine the flow conditions. Introducing the complements α, β, γ of the angles ω, ω, λ' and substituting the values of ψ into expression (5.2), we obtain the equations of the rays AJ, AR, AM.

$$X - \alpha^\circ Y = {}^1/_2\mu_1 + {}^1/_2\alpha^{\circ 2}, \quad X + \beta^\circ Y = \mu_1 + {}^1/_2\beta^{\circ 2}, \quad X - \gamma^\circ Y = {}^1/_2 + {}^1/_2\gamma^{\circ 2} \tag{5.3}$$

Here

$$\alpha^\circ = \alpha / \sqrt{{}^1/_2(n_0 + 1) M_0}, \quad \beta^\circ = \beta / \sqrt{{}^1/_2(n_0 + 1) M_0}, \quad \gamma^\circ = \gamma / \sqrt{{}^1/_2(n_0 + 1) M_0} \tag{5.4}$$

Writing out expressions (5.3) for the point A, we obtain

$$X_A = {}^1/_2 + {}^1/_2\gamma^{\circ 2}, \quad \mu_1 = \alpha^{\circ 2} - \beta^{\circ 2}, \quad \mu_1 = 1 - (\alpha^{\circ 2} - \gamma^{\circ 2}) \tag{5.5}$$

We define the coordinates of the point R as the coordinates of the point where the ray AR is tangent to the arc $\delta = \mu_1$,

$$X_R = X_A - \beta^{\circ 2}, \qquad Y_R = \beta^{\circ}, \qquad (\delta_R = \mu_1) \tag{5.6}$$

We obtain the coordinates of the point M from the condition under which the base of the Mach stem (and therefore the ray AM) is perpendicular to the wall in accordance with (5.2),

$$X_M = X_A - \gamma^{\circ 2}, \qquad Y_M = -\gamma^{\circ} \qquad (\delta_M = {}^1/_2) \tag{5.7}$$

We obtain the conditions for the velocity component v at the points R, M from (5.1),

$$v_R = -\mu_1 (\alpha^{\circ} + \beta^{\circ}), \quad v_M = 0 \tag{5.8}$$

The boundary which closes the short-wave zone downstream coincides with the line of constant velocity $\mu = \mu_1$ of solution (3.3) and contains the sonic line $\delta = \mu_1$ for $\sigma = 0$. The solution $\mu = \delta + Z(\delta)$ then satisfies conditions (4.2),

$$Z(\mu_1) = 0, \qquad Z'(\mu_1) = -1 \tag{5.9}$$

6. The flow beyond the triple point in the short-wave zone is described by the equation

$$-\lambda Z''' + ZZ'' + (Z' + 1)(Z' + Z^1/_2) = 0 \tag{6.1}$$

whose solution must (by (5.6), (5.7), (5.9)) satisfy the conditions

$$Z(\mu_1) = 0, \qquad Z'(\mu_1) = -1, \quad Z({}^1/_2) = {}^1/_2 \tag{6.2}$$

We can determine the range of permissible values of the initial stream parameters α, M_1 by rewriting Eqs. (5.5),

$$\alpha^{*2} - \beta^{*2} = 1, \qquad 1/\mu_1 = 1 + \alpha^{*2} - \gamma^{*2} \tag{6.3}$$

where

$$\alpha^* = \alpha \,/\, \sqrt{{}^1/_2 (n_0 + 1) M_1}, \qquad \beta^* = \beta \,/\, \sqrt{{}^1/_2 (n_0 + 1) M_1}, \qquad \gamma^* = \gamma \,/\, \sqrt{{}^1/_2 (n_0 + 1) M_1} \tag{6.4}$$

As we see from the first equation of (6.3), the initial parameters α and M_1 must satisfy the condition

$$\alpha^* \geqslant 1 \tag{6.5}$$

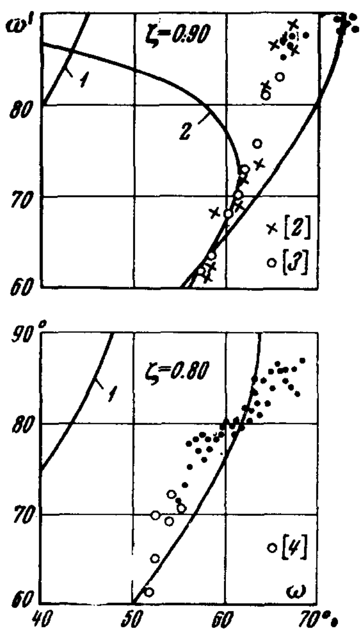

Fig. 2

The dependence of the angle of reflection ω on the angle of incidence ω and on the intensity M_1 is given by the formula

$$\omega' = {}^1/_2\pi - \sqrt{({}^1/_2\pi - \omega)^2 - {}^1/_2 (n_0 + 1) M_1} \tag{6.6}$$

In Fig. 2 we have a comparison of computations using formula (6.6) in the case of low-intensity shock waves ($\zeta = 0.9$, $\zeta = 0.8$, $\zeta = p_0/p_1 = (1 + n_0 M_1)^{-1}$) with the results of the triple-shock (1) and double-shock (2) theories [1, 6] and with the experimental data of various authors [2, 3, 4].

In order to calculate the angle of inclination λ' of the Mach wave (or, which is equivalent, the angle of departure γ) of the triple point from the wall) we must add an additional relation for determining μ_1 to the existing conditions (6.3). This relation can be obtained by using the solution $Z = Z(\delta)$ of Eq. (6.1).

Let us substitute the solution $\mu = \delta + Z(\delta)$ into differential equations (5.2) describing the fronts

$$\frac{d\delta}{dY} = -\sqrt{2\delta - (\mu + \mu_1)}\,, \qquad \frac{d\delta}{dY} = -\sqrt{2\delta - \mu} \tag{6.7}$$

and integrate these equations plotting the fronts of the reflected wave and Mach wave from the points R and M, respectively, to the limiting intersection point S on the ray OA.

Denoting the coordinate of the limit point S by δ^*, we arrive at the expressions

$$\beta^* = \frac{1}{\sqrt{\mu_1}} \int\limits_{\mu_1}^{\delta^*} \frac{d\delta}{\sqrt{\delta - Z(\delta) - \mu_1}}\,, \qquad \gamma^* = \frac{1}{\sqrt{\mu_1}} \int\limits_{\delta^*}^{1/2} \frac{d\delta}{\sqrt{\delta - Z(\delta)}} \tag{6.8}$$

We note that both integrals in (6.8) are convergent despite the singularities at the lower and upper limits in the expressions for β^* and γ^*, respectively.

From now on we shall assume that λ and μ_1 (M_0 and M_1) are given and use Eqs. (6.3), (6.8) to determine α^*, β^*, γ^*.

7. The solutions of differential equation (6.1) satisfying conditions (6.2) were found by numerical integration on a computer for several fixed values of the parameters λ (0, 0.01, 1.0) and μ_1 (0.45, 0.40, 0.33, 0.25, 0.20, 0).

A typical pattern of integral curves $Z = Z(\delta)$ and of the corresponding velocities $\mu = \mu(\delta)$ appears in Fig. 3 ($\mu_1 = 0.25$, $\lambda = 0$, 0.01, 1.0). We see that considerable variation of the velocity (pressure, density, temperature) occurs in the short-wave zone for various $\lambda \neq 0$. These variations are all the more marked the smaller the parameter λ characterizing viscosity and thermal conductivity.

In the limiting case as $\lambda \to 0$ the integral curves approach the vertex of the angle without limit, and then veer off sharply. The solution of Eq. (6.1) for $\lambda \to 0$ differs from the solution for $\lambda = 0$ of the corresponding second-order equation for which the first and second, but not the third, conditions of (6.2) are fulfilled. This is why the ideal-gas model can be used for flows such that $\alpha^* > 1$.

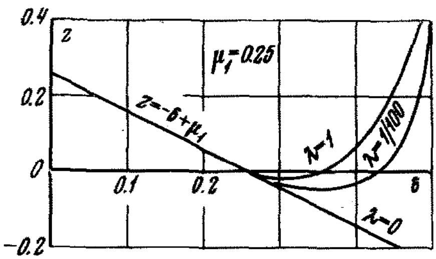

Fig. 3

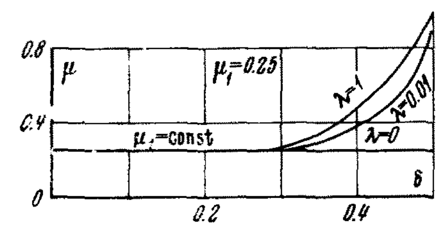

Fig. 4

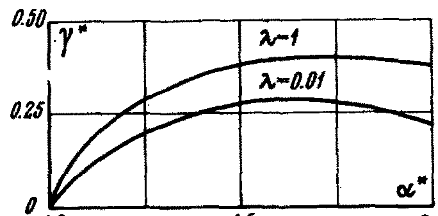

Fig. 5

Another feature of the solutions consists in the fact that as μ_1 tends to $\frac{1}{2}$ for any λ, the short-wave zone contracts to a point on the wall surface; the reflected wave front (which is the sonic line in this case) emerges from the wall at this point. A similar flow pattern arises in the case of an ideal gas ($\alpha^* \leqslant 1$) as $\alpha^* \to 1$ (e.g. see [16]). This leads us to suppose that as $\alpha^* \to 1$ in our case the flow assumes a character typical of a nonviscous gas.

Making use of solution of (6.1) and expressions (6.3), (6.8), we can find the angle of inclination γ^* of the Mach wave. Figure 5 shows the angle γ^* as a function of α^* for various λ. These curves indicate that for $\lambda \neq 0$ and $\alpha^* > 1$ the angle γ assumes small (see (6.4)), but nevertheless nonzero, values.

As $\lambda \to 0$ for all δ different from $^1/_2$ (Figs. 3 and 4) the solution of Eq. (6.1) tends to the limiting form $Z = -\delta + \mu_1$. Computation of integrals (6.8) then yields the relations

$$\beta^* = \sqrt{2/\mu_1}\,\sqrt{\delta^* - \mu_1} \qquad \gamma^* = \sqrt{1/\mu_1}\,(\sqrt{1-\mu_1} - \sqrt{2\delta^* - \mu_1}) \tag{7.1}$$

Substituting results (7.1) into expressions (6.3) and eliminating α^*, we find that $\delta^* = {}^1/_2$ and $\gamma^* = 0$. Thus, in the limiting case $\lambda \to 0$ the angle $\gamma \to 0$, and, by the second expression of (6.3), the maximum relative express pressure at the wall is given by the formula

$$M_0 \,/\, M_1 = 1 \,/\, \mu_1 = 1 + \alpha^{*2} \tag{7.2}$$

The value of $M_0 \,/\, M_1$ as given by formula (7.2) is maximum for all possible values of λ for a fixed α^*.

The computer curves $\gamma^* = \gamma^* (\alpha^*)$ in Fig. 5 can be used to find the function $\gamma = \gamma(\alpha)$ for a given intensity M_1; the curve $\gamma^* = \gamma^* (\alpha^*)$ for $\lambda = 0$ coincides with the α^*-axis. This makes it easy to use formulas (6.3), (6.4), (2.1), (2.2) to find the values of γ and λ for given α and μ_1.

Finally, the solution $\mu = \delta + Z(\delta)$ enables us to use the condition of nonvorticity $\mu_Y = \nu_\delta$ of flow (2.3) to find the transverse velocity component ν satisfying conditions (5.8) of preservation of the tangential velocity components with passage through the wave fronts,

$$\nu = -\frac{\mu_1(\alpha^\circ + \beta^\circ)}{\beta^\circ + \gamma^\circ}(Y + \gamma^\circ) \tag{7.3}$$

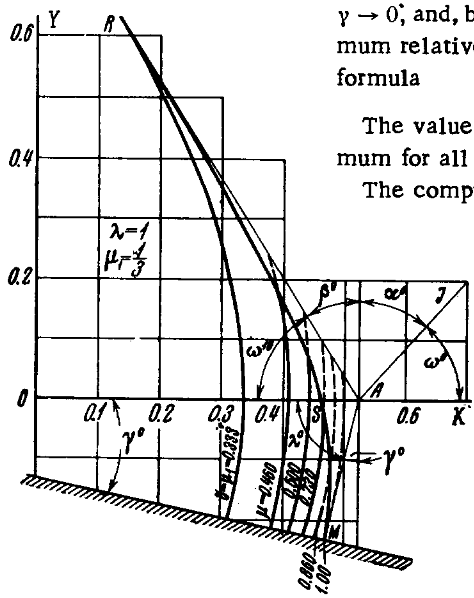

Fig. 6

Figure 6 shows the distribution of the equivelocity lines (isobars) in the flow region beyond the shock fronts in the moving coordinate system X, Y for $\lambda = 1$, $\mu_1 = {}^1/_3$. The shock fronts RS and MS were constructed from the points R and M to their meeting point S on the basis of the results of numerical integration of Eqs. (6.7) under conditions (5.6), (5.7), (6.2).

Since (by virtue of (6.5)) the above flow model is valid for all $\alpha^* \geqslant 1$, increases in the relative excess pressure $M_0 \,/\, M_1$ at the wall with increasing α^* must, by (6.3), ultimately alter the qualitative structure of the stream (i.e. they must bring about a transition to regular reflection, which is possible for $\alpha^* \geqslant 2$ in the case of an ideal gas).

The author is grateful to S. V. Fal'kovich for his suggestions and comments on the present study.

BIBLIOGRAPHY

1. Neumann, J., Oblique reflection of shock. Bourd, U.S. Navy Department, Explosive Res. Rep. №12, 1943.
2. Smith, L. G., Photographic investigations of the reflection of plane shock in air. Office of Scientific Research and Development №6271, 1945.

3. Bleakney, W. and Taub, A. H., Interaction of shock waves. Rev. Mod. Physics Vol. 21. №4, 1949.
4. Kawamura, R. and Saito, H., Reflection of shock waves. 1. Pseudo-stationary case. J. Phys. Soc. Japan Vol. 11, №5, 1956.
5. Fletcher, C. H., Taub, A. H. and Bleakney, W., The Mach reflection of shock waves at nearly glancing incidence. Rev. Mod. Physics Vol. 23, №3, 1951.
6. Pack, D. C., The reflection and diffraction of shock waves. J. Fluid Mech. Vol. 18, №4, 1964.
7. Sternberg, J., Triple shock waves intersection. Phys. Fluids Vol. 2, №2, 1959.
8. Sakurai, A., On the problem of weak Mach reflection. J. Phys. Soc. Japan Vol. 19, №8, 1964.
9. Shao, T. S., Numerical solution of plane viscous shock reflections. J. Comput. Physics Vol. 1, №3, 1967.
10. Grib, A. A., Ryzhov, O. S. and Khristianovich, S. A., A theory of short waves. Prik. Mekh. i Tekh. Fiz №1, 1960.
11. Ryzhov, O. S. and Shefter, G. M., On the effect of viscosity and thermal conductivity on the structure of compressible flows. PMM Vol. 28, №6, 1964.
12. Sichel, M., The effects of longitudinal viscosity on the flow at a nozzle throat. J. Fluid Mech. Vol. 25, №4, 1966.
13. Ryzhov, O. S., Some problems of transonic flows of a real gas. Zh. Vychisl. Mat. i Matem. Fiz. Vol. 8, №2, 1968.
14. Tomotika, S. and Tamada, K., Studies on two-dimensional transonic flow of compressible fluids. Part 1. Quart. Appl. Math. Vol. 7, №4, 1950.
15. Landau, L. D. and Lifshits, E. M., Mechanics of Continuous Media, 2nd edition. Gostekhizdat, Moscow, 1953.
16. Shindiapin, G. P., On the irregular reflection of weak shock waves from a rigid wall. Prik. Mekh. i Tekh. Fiz. №2, 1964.

Translated by A. Y.

ON THE GENERALIZED ORTHOGONALITY RELATION OF P. A. SCHIFF

PMM Vol. 33, №2, 1969, pp. 376-383

B. M. NULLER
(Leningrad)

(Received June 25, 1968)

It has been assumed until now that Papkovich [1, 2] was the first author to derive the generalized orthogonality relation and to pose the problem of simultaneous expansion of two independent functions in series in homogeneous solutions. This problem has been dealt with within the framework of the plane problem of elasticity theory by Grinberg [3], Prokopov [4], Vorovich and Koval'chuk [5], and by several foreign authors whose studies are summarized in survey [6].

However, as was recently discovered, Papkovich's paper [1] gave impetus to studies of a problem whose history dates back to a variant of the three-dimensional problem of the theory of elasticity. We are referring to a paper by Schiff [8] (1883) which contains a